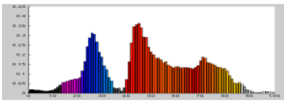


Dr. Qadri Hamarsheh

Signals and Systems

Lecture 6

Linear Time-Invariant Systems
(LTI Systems)

Outline

- **Basic System Properties**
  - ✓ **Memoryless and systems with memory (static or dynamic).**
  - ✓ **Causal and Non-causal systems (Causality).**
  - ✓ **Linear and Non-linear systems (Linearity).**
  - ✓ **Stable and Non-stable systems (Stability).**
  - ✓ **Time-Invariant systems (Time invariance).**

Basic System Properties

1. Memoryless and systems with memory (static or dynamic).
2. Causal and Non-causal systems (Causality).
3. Linear and Non-linear systems (Linearity).
4. Stable and Non-stable systems (Stability).
5. Time-Invariant systems (Time invariance).

Memoryless and systems with memory (static or dynamic):

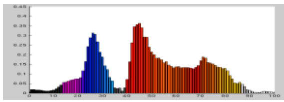
A system is called **memoryless**, if the output, $y(t)$, of a given system for each value of the independent variable at a given time t depends only on the input value at time t.

A system has **memory** if the output at time t_1 depends in general on the past values of the input $x(t)$ for some range of the values of t to $t = t_1$. A system with memory retains or stores information about input values at times other than the current input value.

> **D-T signal terms:**
> The transformation does not depend on the previous samples of the sequence, it is memoryless D-T system

Examples:

System Name	System Equation Definition	Description
Ideal Amplifier/Attenuator	$y(t) = k \cdot x(t)$, $y[n] = k \cdot x[n]$ where k is some real constant	Memoryless
Integrator	$y(t) = \int_{-\infty}^{t_1} x(t)dt$ Integrate the values of the input signal from all past times up to present time.	System with memory



Examples:

1. Continuous time systems:

a) $y(t) = 5\sin(t)\cdot\cos(3t)$: This system is memoryless.

b) $y(t) = \int_{-\infty}^{t/7} x(\tau)d\tau$

For some general input function $x(t)$, this system is a system with memory, because it depends on all past values of the input.

c) $y(t) = \int_{t_0}^{t} \tau\cdot e^{-\tau}d\tau$ - consider $x(t) = t\cdot e^{-t}$

Solution: we use integration by parts,

$$u = \tau \Rightarrow du = d\tau$$

$$dv = e^{-\tau} \Rightarrow v = -e^{-\tau} \Rightarrow$$

$$u\cdot v - \int vdu = -\tau\cdot e^{-\tau}\Big|_{t_0}^{t} + \int_{t_0}^{t} e^{-\tau}d\tau =$$

$$-\tau\cdot e^{-\tau}\Big|_{t_0}^{t} - e^{-\tau}\Big|_{t_0}^{t} = t\cdot e^{-t} + t_0\cdot e^{-t_0} - e^{-t} + e^{-t_0}$$

$$y(t) = e^{-t_0}(1+t_0) - e^{-t}(1+t)$$

So we have system with memory.

2. Discrete time systems :

a. $y[n] = x[n-5]$

This system in not memoryless, because the output value at n depends on the input values at $n-5$

b. $y[n] = \sin(x[n]) + 5$ - memoryless system.

Causal and Non-causal systems:

If the output of the system $y(t)$ at any time depends only on the input at present and/or previous times, we say that the system is causal, mathematically this can be represented as $y(t) = f(x(t), x(t-1), \ldots)$.

A noncausal system anticipates the future values of the input signal in some way.

All memoryless systems are causal

For real time system where n actually denoted time causality is important. Causality is not an essential constraint in applications where n is not time, for example, image processing. If we are doing processing on recorded data, then also causality may not be required.

Examples:

1. Continuous time systems:

a) **Ideal Predictor**: this system is given by the following input-output relationship $y(t) = x(t+1)$, it is noncausal system, since the value $y(t)$ of the output at time t depends on the value $x(t+1)$ of the input at time $t+1$, so the output must appear before the input signal as shown in figure 3-7.

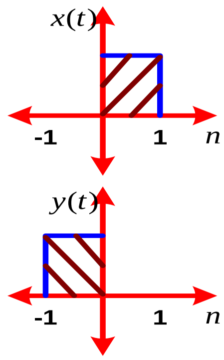


Figure 3-7: Ideal Predictor

In general, $y(t) = k \cdot x(t+q)$, where q is a positive real number, is a noncausal system.

b) Ideal Time Delay :

The Ideal Time Delay has the following equation $y(t) = x(t-1)$ and this system is causal.

2. Discrete time systems :

a) $N-$point MA Filter:

The $N-$point MA Filter $y[n] = \frac{1}{N}\left[x[n] + x[n-1] + x[n-2] + ... + x[n-N+1]\right]$ is a causal system.

b) $9-$point MA Filter:

The 9-point MA Filter with the following definition:

$$y[n] = \frac{1}{9}\left[x[n+4] + x[n+3] + x[n+2] + x[n+1] + x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\right]$$

is a noncausal filter, since the filter output at time n requires the future values $x[n+4], x[n+3], x[n+2]$ and $x[n+1]$ of the input.

c) The system defined by $y[n] = \frac{1}{2N+1}\sum_{k=-N}^{N} x[n-k]$ is noncausal.

d) $y[n] = 3x[n-3]$ - is a causal, since the output value at n for the system described by $y[n] = 3x[n-3]$ depends on the previous values of n.

e) $y[n] = 5x^3[n+3]$ - is a noncausal, since the output value n depends on the input value $n+3$.

Linear and Non-linear systems (Linearity):

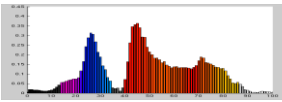
This is an important property of the system. We will see later that if we have system which is linear and time invariant then it has a very compact representation. An operator $\hat{T}$ is called linear if the following relationships hold:

$$\hat{T}\{ax_1(t) + bx_2(t)\} = a \cdot \hat{T}\{x_1(t)\} + b \cdot \hat{T}\{x_2(t)\} = a \cdot y_1(t) + b \cdot y_2(t) \text{ for C-T signals.}$$

Or

$$\hat{T}\{ax_1[n] + bx_2[n]\} = a \cdot \hat{T}\{x_1[n]\} + b \cdot \hat{T}\{x_2[n]\} = a \cdot y_1[n] + b \cdot y_2[n] \text{ for D-T signals.}$$

To explain these relationships, suppose that $\hat{T}$ acts on two input signals $x_1(t)$ and $x_2(t)$ to produce the following signals:



$y_1(t) = \hat{T}\{x_1(t)\}$ - The response of the system to the input $x_1(t)$.

and $y_2(t) = \hat{T}\{x_2(t)\}$ - The response of the system to the input $x_2(t)$.

And suppose that a and b are two constants. To get the linearity property, a linear system has the important property of **superposition**:

> superposition
>
> If an input consists of weighted sum of several signals ($ax_1(t)+bx_2(t)$), the output is also weighted sum of the responses of the system to each of those input signals ($a \cdot y_1(t)+b \cdot y_2(t)$).

The **superposition** property consists of two parts:

- **Additivity:** The response to $\{x_1(t)+x_2(t)\}$ is $\{y_1(t)+y_2(t)\}$.
- **Homogeneity:** The response to $a\{x_1(t)\}$ is $a\{y_1(t)\}$, where a is any real number if we are considering only real signals and a is any complex number if we are considering complex valued signals. This means that **if a system is homogeneous, then the scaled input gives a scaled output** (some scaling factors).

From figure 3-8, we see that we can decompose complicated signal $x(t)$ into a sum of simpler signals $x_1(t)$ and $x_2(t)$, and then treat each of these signals through the system.

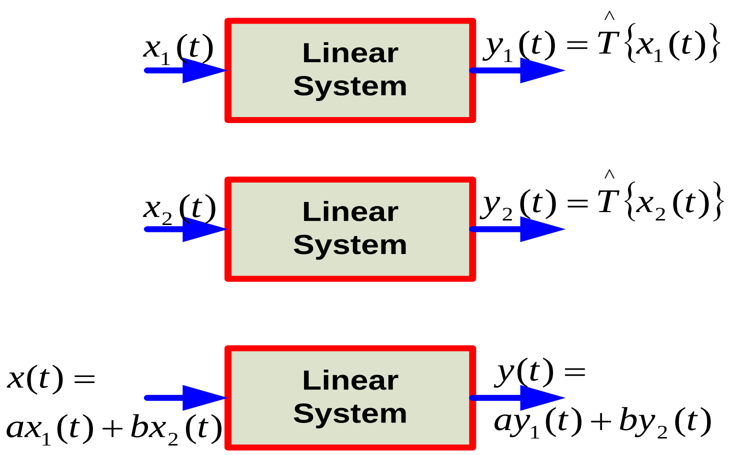


Figure 3-8: Linearity

System linearity checks

To determine that the system is linear, use the following steps (see also the figure 3-8):

1. Form the sum $ay_1(t)+by_2(t)$ considering two input-output relationship $y_1(t)$ and $y_2(t)$.
2. Construct the response, $\hat{T}\{ax_1(t)+bx_2(t)\}$, of the input: $ax_1(t)+bx_2(t)$.
3. Check for equality the response of step 1 with the response of step 2, if these two responses are equal, then the system is linear.

Examples:

1. **Continuous time systems:**

   a) $y(t) = 5x(t)$

Solution:

From step 1, we consider two inputs and output signals multiplied by scalars. Since $y(t) = 5x(t)$, we have

$$y_1(t) = 5x_1(t),$$
$$y_2(t) = 5x_2(t)$$

and the sum weighted by two constants is:

$$ay_1(t) + by_2(t) = 5ax_1(t) + 5bx_2(t).$$

From step 2, we use the sum

$ax_1(t) + bx_2(t)$ as an input to the system, where

$$\hat{T}\{ax_1(t) + bx_2(t)\} = 5 \cdot [ax_1(t) + bx_2(t)] = 5ax_1(t) + 5bx_2(t).$$

From step 3, these equations are equal, and then the system is **linear**.

   b) $y(t) = Rx^3(t)$, where **R** is a constant.

Solution:

From step 1, we have:

$$ay_1(t) + by_2(t) = aRx_1^3(t) + bRx_2^3(t)$$

From step 2, we consider the transformation acting on $ax_1(t) + bx_2(t) \Rightarrow$

$$\hat{T}\{ax_1(t) + bx_2(t)\} = R \cdot (ax_1(t) + bx_2(t))^3$$

Using $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ we obtain the following equation

$$\hat{T}\{ax_1(t) + bx_2(t)\} = R \cdot (ax_1(t) + bx_2(t))^3 = R \cdot \left(a^3x_1^3(t) + 3a^2bx_1^2(t)x_2(t) + 3ab^2x_1(t)x_2^2(t) + b^3x_2^3(t)\right)$$

From step 2, the system is **not linear**.

   c) $y(t) = \dfrac{d^2x}{dt^2}$

Solution:

$$ay_1(t) + by_2(t) = a\frac{d^2x_1}{dt^2} + b\frac{d^2x_2}{dt^2}$$

Now

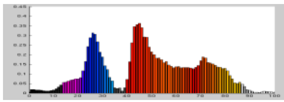
$$\hat{T}\{ax_1(t) + bx_2(t)\} = \frac{d^2[ax_1(t) + bx_2(t)]}{dt^2} = a\frac{d^2x_1(t)}{dt^2} + b\frac{d^2x_2(t)}{dt^2} \Rightarrow$$

in this case, the system is **linear**, as the result of the first step and second step are equal.

2. **Discrete time systems:**

   a) $y[n] = x[n-5]$

$\hat{T}\{ax_1[n] + bx_2[n]\} = ax_1[n-5] + bx_2[n-5] = ay_1[n] + by_2[n] \Rightarrow$ therefore $y[n] = x[n-5]$ is linear system



b) $y[n] = x[n]u[n-k], k > 0$

$\hat{T}\{ax_1[n] + bx_2[n]\} = ax_1[n]u[n-k] + bx_2[n]u[n-k] = ay_1[n] + by_2[n] \Rightarrow$ linear system

Stable and Non-stable systems (Stability):

There are several definitions for stability. Here we will consider bounded input bonded output (**BIBO**) stability. A system is said to be BIBO stable if every bounded input produces a bounded output.

We say that a signal $x[n]$ is bounded if we can find a constant M such that for all n, $|x[n]| < M < \infty$, and we say that the output signal $y[n]$ is also bounded if we can find a constant K such that $|y[n]| < K < \infty$.

Examples:

- **a)** The moving average system defined by $y[n] = \frac{1}{2N+1}\sum_{k=-N}^{N} x[n-k]$ is stable as $y[n]$ is sum of finite numbers and so it is bounded.
- **b)** The accumulator system defined by $y[n] = \sum_{k=-\infty}^{n} x[k]$ is unstable. If we take $x[n] = u[n]$, the unit step then $y[0] = 1, y[1] = 2, y[2] = 3, \ldots y[n] = n+1, \quad n \geq 0$ so $y[n]$ grows without bound.
- **c)** $y[n] = 7x[n-3]$, assume that, $x[n] \leq M$ for some finite M for all n.

  In this case $x[n] \leq M$ implies that $|y[n]| \leq 7M$, so the system is stable.
- **d)** $y[n] = 2nx[n-1]$, assume that, $x[n] \leq M$ for some finite M for all n.

In this case, since we have $y[n] = 2nx[n-1]$, were the output directly depends on n, it grows without bound as n increases, so the system is not stable.

Time-Invariant systems (Time invariance):

A system is said to be time invariance (**TI**) if the behavior and the structure of the system do not change with time (TI system responds exactly the same way no matter when the input signal is applied). Thus a system is said to be time invariant if a time shift (delay or advance) in the input signal, $x(t) \rightarrow x(t \pm \tau)$, leads to identical delay or advance in the output signal. Mathematically

If $y[n] = \hat{T}\{x[n]\}$ then $y[n-n_0] = \hat{T}\{x[n-n_0]\}$ for any n_0

Time invariance testing steps (see figure 3-9):

1. Find the shifted output $y[n-n_0]$ of the system.
2. Find the output of the shifted input $y_{n0}[n] = \hat{T}\{x[n-n_0]\}$.
3. Compare step1 and step 2, if they are equal, then the system is time invariance.

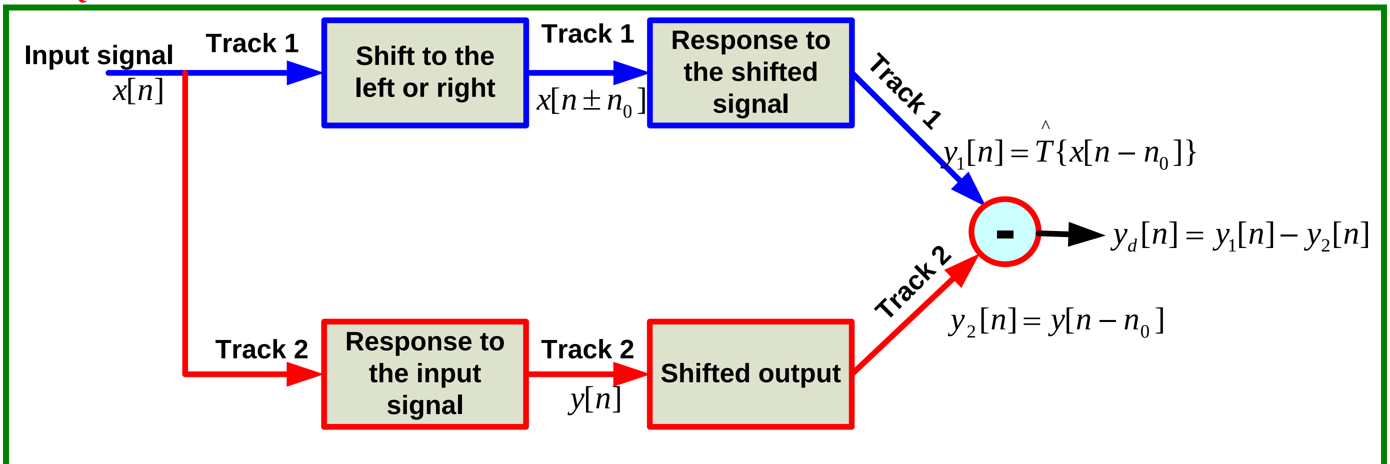


Figure 3-9: Time Invariance Testing steps

Examples:

1. Continuous time systems:

Determine whether or not the system is a time-invariant for t_0?

$$y(t) = tx(t)$$

To solve this task, we go through the steps described above for time invariance testing:

Step1: the result of the shifted output:

$$y(t-t_0) = (t-t_0)x(t-t_0)$$

Step 2: the output of the shifted input:

$$y_\tau(t) = \hat{T}\{x(t-t_0)\} = tx(t-t_0)$$

Step 3: Comparing the two outputs we see that they are not equal, so this system is **time varying**.

2. Discrete time systems:

Determine whether or not the systems are time-invariant?

a. $y[n] = 7x[n-7]$.

Solution:

Step1: the result of the shifted output:

$$y[n-n_0] = 7x[n-n_0-7]$$

Step 2: the output of the shifted input:

$$y_{n0} = \hat{T}\{x[n-n_0]\} = 7x[n-n_0-7]$$

Step 3: Comparing the two outputs we see that they are equal, so this system is **time invariant**.

b. $y[n] = 7nx[n]$

Solution:

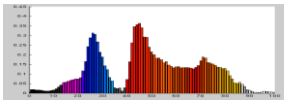
Step1: the result of the shifted output:

$$y[n-n_0] = 7(n-n_0)x[n-n_0]$$

Step 2: the output of the shifted input:

$$y_{n0} = \hat{T}\{x[n-n_0]\} = 7nx[n-n_0]$$

Step 3: Comparing the two outputs we see that they are not equal, so this system is **time varying**.



c. Accumulator system : $y[n]=\sum_{k=-\infty}^{n} x[k]$

Solution:

Step1: the result of the shifted output is given by:

$$y[n-n_0]=\sum_{k=-\infty}^{n-n_0} x[k]$$

Step 2: the output of the shifted input:

$y_{n0}=\hat{T}\{x[n-n_0]\}$, let $x_1[n]=x[n-n_0]$, then the corresponding output is

$$y_1[n]=\sum_{k=-\infty}^{n} x_1[k]=\sum_{k=-\infty}^{n} x[k]$$

Step 3: Comparing the two outputs:

The two expressions are equal. To get that, let us change the index of summation by $i=k-n_0$ in the second sum then we see that

$$y_1[n]=\sum_{i=-\infty}^{n-n_0} x[i]=y[n-n_0]$$

So this system is **time invariant**.

A system can be linear without being time invariant and it can be time invariant without being linear.